MATH 579 Exam 8 Solutions

Part I: An L-triomino is a tile shaped like a 2×2 grid with one corner missing. A monomino is a single 1×1 square. How many ways there are to tile a $2 \times n$ chessboard with L-triomino's and monomino's?

> Let a_n represent the desired quantity. We have $a_0 = a_1 = 1, a_2 = 5$. A 'break' is a line (the short way) across the strip that does not intersect any of the tiles. The first break can happen at position 1, 2, or 3. If at position 1, there is one way to tile the 2×1 piece, and a_{n-1} ways to tile the remainder. If at position 2, there are four ways to tile the 2×2 piece, and a_{n-2} ways to tile the remainder [Note: only four, not five, because this is the *first* break.]. If at position 3, there are two ways to tile the 2×3 piece, and a_{n-3} ways to tile the remainder. This leads to the recurrence relation $a_n = a_{n-1} + 4a_{n-2} + 2a_{n-3}$. This has characteristic equation $s^3 - s^2 - 4s - 2 = 0$, which has roots $-1, 1+\sqrt{3}, 1-\sqrt{3}$. Applying the initial conditions gives the specific solution $a_n = (-1)^n + \frac{\sqrt{3}}{3}\left((1+\sqrt{3})^n - (1-\sqrt{3})^n\right)$.

Part II:

1. Solve the recurrence $a_0 = 2, a_n = 3a_{n-1}$ $(n \geq 1)$.

   This has characteristic equation $s - 3 = 0$, so the general solution is $a_n = A3^n$. The initial conditions give $2 = a_0 = A3^0$, so $A = 2$ and $a_n = 2 \cdot 3^n$.

2. Solve the recurrence $a_0 = 2, a_n = 3a_{n-1} - 2$ $(n \geq 1)$.

   As per the previous problem, the homogeneous solution is $a_n = A3^n$. To solve the nonhomogeneous recurrence, we guess the constant polynomial $a_n = B$, so $B = 3B - 2$ and hence $B = 1$. Hence the general nonhomogeneous solution is $a_n = A3^n + 1$. The initial condition gives $2 = a_0 = A + 1$, so $A = 1$ and $a_n = 3^n + 1$.

3. Solve the recurrence $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + e^n$ $(n \geq 2)$.

   The characteristic equation is $s^2 - s - 2 = 0$, which has roots $2, -1$ so the general homogeneous solution is $a_n = A2^n + B(-1)^n$. We guess $A_n = Ce^n$ for the nonhomogeneous case, getting $Ce^n = Ce^{n-1} + 2Ce^{n-2} + e^n$. Dividing by e^{n-2} we get $Ce^2 = Ce + 2C + e^2$, or $C(e^2 - e - 2) = e^2$. Hence $a_n = A2^n + B(-1)^n + \frac{e^2}{e^2-e-2}e^n$ is the general solution to the nonhomogeneous problem. The initial conditions and a bit of algebra gives $a_n = \frac{-e^2}{3(e-2)}2^n + \frac{e^2}{3(e+1)}(-1)^n + \frac{e^2}{(e-2)(e+1)}e^n = C((e+1)2^n + (e-2)(-1)^n)/3 + Ce^n$.

4. My credit card charges 18% interest, compounded monthly. I make a \$1000 purchase, and make only my \$25 minimum payment each month. Find the balance on my card after n months.

   Let b_n denote the card balance. The problem specifies that $b_n = (1+\frac{0.18}{12})b_{n-1} - 25 = 1.015b_{n-1} - 25$, and $b_0 = 1000$. The general homogeneous solution is $b_n = A(1.015)^n$, and we seek a nonhomogeneous solution via $b_n = B$: $B = 1.015B - 25$, so $0.015B = 25$ and $B = \frac{5{,}000}{3}$. Hence the general solution is $b_n = A(1.015)^n + \frac{5{,}000}{3}$; the initial conditions give $1000 = b_0 = A + \frac{5{,}000}{3}$ so $A = -\frac{2{,}000}{3}$ and $b_n = \frac{1000}{3}(-2(1.015)^n + 5)$. As a side note, using logarithms we can calculate that the debt will be paid off (b_n negative) after 62 months, costing a total of almost \$3100.

5. We color each square of a $1 \times n$ chessboard with a color chosen from $[m]$ $(m > 1)$, with the rule that the single color m may not be used twice in a row. How many ways can we do this?

   Let a_n denote the desired quantity; we have $a_0 = 1, a_1 = m$. If the first square is colored anything but m, the remainder can be colored in a_{n-1} ways. If the first square is colored m, then the next square must be colored something else ($m-1$ choices), and the balance can be colored in a_{n-2} ways. Hence a_n satisfies the recurrence $a_n = (m-1)a_{n-1} + (m-1)a_{n-2}$. This has characteristic equation $s^2 - (m-1)s - (m-1) = 0$, and roots $\alpha = \frac{m-1+\sqrt{m^2+2m-3}}{2}, \beta = \frac{m-1-\sqrt{m^2+2m-3}}{2}$. Using the initial conditions we find $a_n = \frac{m-\beta}{\alpha-\beta}\alpha^n + \frac{\alpha-m}{\alpha-\beta}\beta^n$.

Exam grades: High score=100, Median score=78, Low score=52